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Evaluating Time Streams of Income

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Evaluating Time Streams of Income*

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Abstract

When a decision maker considers possible returns from a business project or investment, he often faces the problem that these returns are not all received at the same time, and thus he must make some adjustments to take account of his time preference for money. After a review of discounting, a utility theory approach is made by developing a two-attribute utility function u(x,t) which represents the desirability of an income of x at a time t in the future. Assumptions to simplify the assessment of this function are considered. Then u(x,t) is used to form a criterion for evaluating infinite time streams of income.

When a decision maker considers possible returns from a business project or investment, he often faces the problem that these returns are not all received at the same time, and thus he must make some adjustments to take account of his time preference to money.

This paper uses utility theory to examine the problem of evaluating time streams of income both in circumstances of certainty and uncertainty with regard to the exact value and timing of the incomes.

If a comparison of value between two sums of money is to be made, where one sum is expressed in dollars, the other in pounds, the first step would be to convert one sum into the units of the other.

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A similar situation arises when making comparisons between two sums of money offered at different times. Compare an offer of \$100 to be received now with one of \$120 to be received in one year's time. If both offers were for the same time period there would be no problem in making a choice, but the time lag of one year in the more valuable offer makes the preference of \$120 less likely. Unlike the existence of exchange rates for foreign currencies there is no easy table for calculating equivalences of cash between different time periods.

There is, however, the facility to lend and borrow money at fixed interest rates at banks and similar institutions.

Suppose that we have a means of earning 100i% interest per annum on an investment, then in our example \$100 invested now will be worth \$100(1 + i) after one year. So it is worthwhile considering whether $100(1 + i) > 120$. For if so, then it is evidently wise to prefer the \$100 now to the \$120 in one year.

Suppose also that we have a means of borrowing money for any given length of time to be repaid with a compound charge of 100r% per annum on the loan. So we could borrow $\$\frac{120}{1 + r}$ now and when we receive the \$120, pay back the principal and the interest. So, is $\frac{120}{1 + r} > 100$? If so, then we should prefer the \$120 offer.

This simple rule will never be contradictory as long as $r \geq i$ which is the case to be expected; otherwise we could make a large profit by reinvesting loans.

If the simplifying assumption that i = r is made (termed an infinite linear bank) then an amount \$A to be received at time T must be preferred to an amount \$B at time S if and only if

$$\frac{A}{(1 + r)^T} \geq \frac{B}{(1 + r)^S} \quad .$$

This procedure of comparison is known as discounting and r is the discount rate. To implement this method requires only the assessment of a value of r suitable to the decision maker.

However, the situation i = r is idealistic and Figure 1 shows the situation when r > i. An amount \$x to be received at time t will be denoted (x,t), and if (x_1,t_1) is considered indifferent by the decision maker in question to (x_2,t_2) we will write

$$(x_1,t_1) \sim (x_2,t_2) \quad .$$

The shaded area in Figure 1 represents all the points of the x-t plane which could be indifferent to \$100. If the point (120,1) lies in this shaded area the question of preference between (100,0) and (120,1) is unresolved and is a matter of personal time preference for the decision maker.

The decision maker may wish to maximize the money available to him now or he may wish to raise the most capital for some venture in the future. It is this personal time preference which is unaccounted for when present value discounting

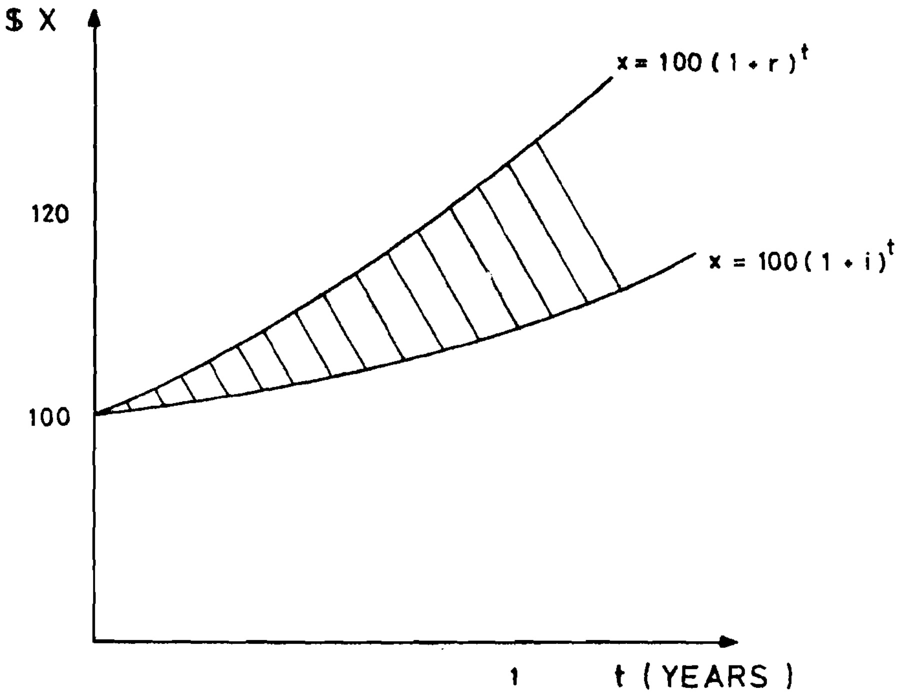

FIGURE 1. THE EXISTENCE OF TIME PREFERENCE.

is used and it is the aim of this paper to provide a scheme to incorporate this preference into the decision procedure.

We will not attempt to give here a grounding in utility theory (see Raiffa [7] instead). But the essence of a utility function, say for money, $u^*(x)$, is that for a situation having uncertain outcomes, a probability expectation of the function will produce a certainty equivalent which incorporates the decision maker's attitude to risk.

For our problem let us introduce a two-dimensional, or two-attribute, utility function $u(x,t)$ which represents the value to the decision maker of an extra income of \$x to be received at time t but promised now ($t = 0$). That is, the money is in addition to all presently perceived income. To demonstrate the usefulness of a utility function, consider a firm which is offered a project which has a 50-50 chance of success with a profit of one million dollars. Unfortunately failure will mean a loss of \$90,000. The firm we are considering faces bankruptcy (or worse) if its debts rise to \$100,000. The results of the project will not be known for one year. Should the firm accept the project?

Discounting at, say, $r = 0.1$, yields a present value of $\frac{1}{2} \cdot \frac{10}{11} \cdot [1{,}000{,}000 - 90{,}000] = +\$410{,}000$, indicating acceptance. A possible utility function for the firm might be

$$u(x,t) = (\tfrac{10}{11})^t \log (x + 100{,}000) \quad .$$

The certainty equivalent of the project c is given by

$$u(c,0) = \frac{1}{2}\,u(1{,}000{,}000,1) + \frac{1}{2}\,u(-90{,}000,1) \quad ,$$

which gives a value for c of +$36,000.

Both systems recommend acceptance but the utility function has already reduced the present value of the gamble to reflect the risk averseness of the firm to gambles which involve possible large debts.

The example is exaggerated for effect but demonstrates the ideas involved.

The Calculation of the Utility Function

The theory of utility is extremely useful--in theory, but its flaw lies in that implementing the theory can be difficult in practice. The difficulty lies in assessing the utilit functions required; the more attributes a utility function has, the more complicated the assessment. For a simple problem with relatively small amounts of cash involved it may well not be worth going through the trouble of actually assessing the function. However, for more weighty decisions, or for regular financial decisions, it may well be worth investing the time in assessment.

The utility function in question may be assessed directly as a two-attribute function, but if a simplifying assumption can be found the assessment will be easier. We will assume that a utility function for money alone, $u^*(x)$, has already been calculated and scaled so that $u^*(0) = 0$. Note that $u^*(x) = u(x,0)$.

Let us consider three possible assumptions and their simplifying effects.

1. Weak Stationarity of Time Preferences

Consider an uncertain situation which will result in either a payoff or \$x or \$y with equal probability, both amounts to be received at time t. Suppose that it is felt that an amount \$z for sure, to be received at time t, is just equivalent in value to the gamble. That is

$$u(z,t) = \frac{1}{2}\, u(x,t) + \frac{1}{2}\, u(y,t) \quad . \tag{1}$$

The assumption is as follows. Suppose that the value of t were altered, would this alter the value of z? If not then we can say that (1) is true for all values of t.

If this assumption is true then we can say (see Keeney [2]) that

$$u(x,t) = f(t) + g(t)\, u(x,T)$$

for some functions f, g and for any value T.

Clearly $(0,t)\sim(0,0)$ so that $u(0,t) = u(0,0) = u^*(0) = 0$; by substituting $T = 0$, $x = 0$ into (1) we see that $f(t) = 0$. Hence

$$u(x,t) = g(t)\, u(x,0) = g(t)\, u^*(x) \quad .$$

Thus, $u(x,t)$ is known after the assessment of a one-dimensional time function $g(t)$, a much easier task.

2. Strong Stationarity of Time Preferences

This second assumption implies the first so cannot hold if the first does not. The first assumption considered gambles where all payoffs were at the same time. Suppose that in a 50-50 gamble between (x,t) and (y,s) the decision maker feels (z,r) for certain is just equivalent. Then if the whole gamble is delayed an amount h in time, can we assert that (z,r + h) is just equivalent to the delayed lottery?

If so, then since $u(x,t) = g(t)\, u^*(x)$ from the first assumption, we have that

$$g(r + h)\, u^*(z) = \frac{1}{2}\, g(t + h)\, u^*(x) + \frac{1}{2}\, g(s + h)\, u^*(y) \qquad (2)$$

for all h. Put y = 0 as a special case, then (2) becomes

$$\frac{g(r + h)}{g(t + h)} = \frac{u^*(x)}{2u^*(z)}$$

for all h. So

$$\frac{g(r + h)}{g(t + h)} = \text{constant}.$$

Let $h = -\min(r,t) = -r$ say, so that

$$\frac{g(r + h)}{g(t + h)} = \frac{1}{g(t - r)} \qquad (\text{since } g(0) = 1) \quad .$$

Letting $t - r = m$ and $r + h = n$, we have

$$g(m + n) = g(m)\, g(n)$$

from which we deduce that

$$g(t) = e^{-ct} \qquad \text{for some } c \quad .$$

Hence,

$$u(x,t) = e^{-ct}\, u^*(x) \quad ,$$

or

$$u(x,t) = \frac{u^*(x)}{(1 + r)^t} \quad ,$$

where $1 + r = e^c$. This is called utility discounting and specializes to the case of ordinary discounting if $u^*(x)$ is assumed to be linear.

3. Temporal Invariance of Indifference

This last assumption considered here is the weakest of the three. Its implications are not precise but it is presented here because the assumption may be often more readily applicable than the previous two: If $(x,t) \sim (y,s)$ then $(x,t + h) \sim (y,s + h)$ for all $h \geq 0$. That is, if two quantities are considered equivalent and are then delayed by equal amounts they remain indifferent to each other.

The effect on the form of the utility function can be observed by examination of Figure 2. The curve $x = f_1(t)$ represents all those points in the (x,t) plane which are indifferent to $(1,0)$. Similarly $x = f_2(t)$ represents all those points indifferent to $(2,0)$. Suppose that for some particular values of x and t, $(2,0) \sim (x,t)$. For some time value s,

$$(1,0) \sim (2,s) \quad ,$$

and by the assumption,

$$(2,s) \sim (x,t + s) \quad .$$

So

$$x = f_2(t) => x = f_1(t + s)$$

or

$$f_2(t) = f_1(t + s) \quad .$$

In general,

$$f_x(t - c) = f_y(t) \quad ,$$

where $(x,c) \sim (y,0)$. (3)

So, suppose that we calculate an indifference curve $x = f(t)$ or $u(x,t) =$ constant. By consideration of Figure 3 and (3) we can see that $u(x,t)$ is completely determined in the shaded region. In fact, $u(x,t) = u^*[f(f^{-1}(x) - t)]$ for $x \geq f(t)$. A similar analysis yields appropriate results for the case $x < 0$.

For example, if $u^*(x) = 1-e^{-cx}$ and $(1,0) \sim (e^t,t)$ for all t, then $u(x,t) = 1 - \exp[-c \exp(\log x - t)] = 1 - \exp(-cxe^{-t})$

The Evaluation of Time Streams

We have shown how a money utility function $u^*(x)$ may be extended to a utility function $u(x,t)$ dealing with money and time. A more difficult and perhaps insurmountable problem is that of extending it further to deal with time streams of in-

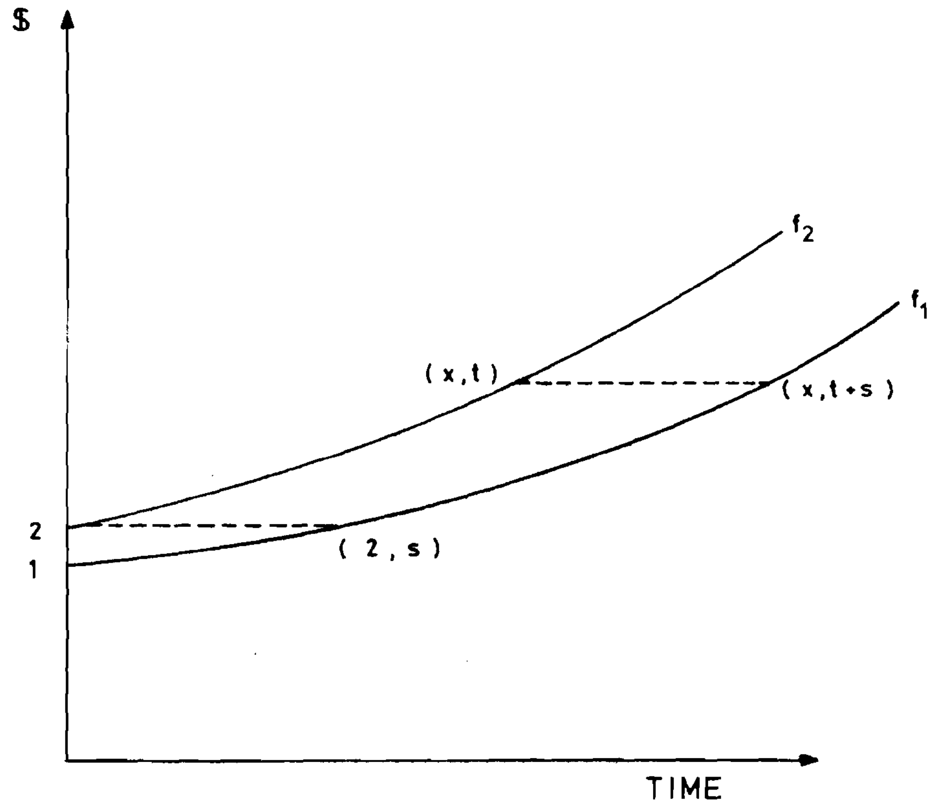

FIGURE 2. THE IMPLICATIONS OF ASSUMPTION 3.

come. A time stream may be represented discretely as an infinite vector $(x_o,x_1,x_2,x_3,\ldots)$ where x_i represents the income to be received in period i, or continuously as a function x(t) which represents the total net income from the stream at time t.

To make clear the problem involved in finding a utility function over time streams consider the case of a joint income of (x,t) and (y,s).

A first guess would probably be to assign a utility of

$$\bar{u}\ \{(x,t) + (y,s)\} = u(x,t) + u(y,s)$$

to this double income. But this would imply that

$$\bar{u}\ \{(x,t) + (x,t)\} = u(x,t) + u(x,t)$$

or

$$u(2x,t) = 2u(x,t) \quad .$$

This will only be the case if u(x,t) = xf(t) for some function f. Note that discounting has a utility function of this form, namely

$$u(x,t) = \frac{x}{(1 + r)^t} \quad ,$$

so that the utility of a time stream is just the sum of the utilities of its components,

$$\bar{u}\ (x_o,x_1,x_2,\ldots) = \sum_{k=0}^{\infty} u(x_k,k) \quad .$$

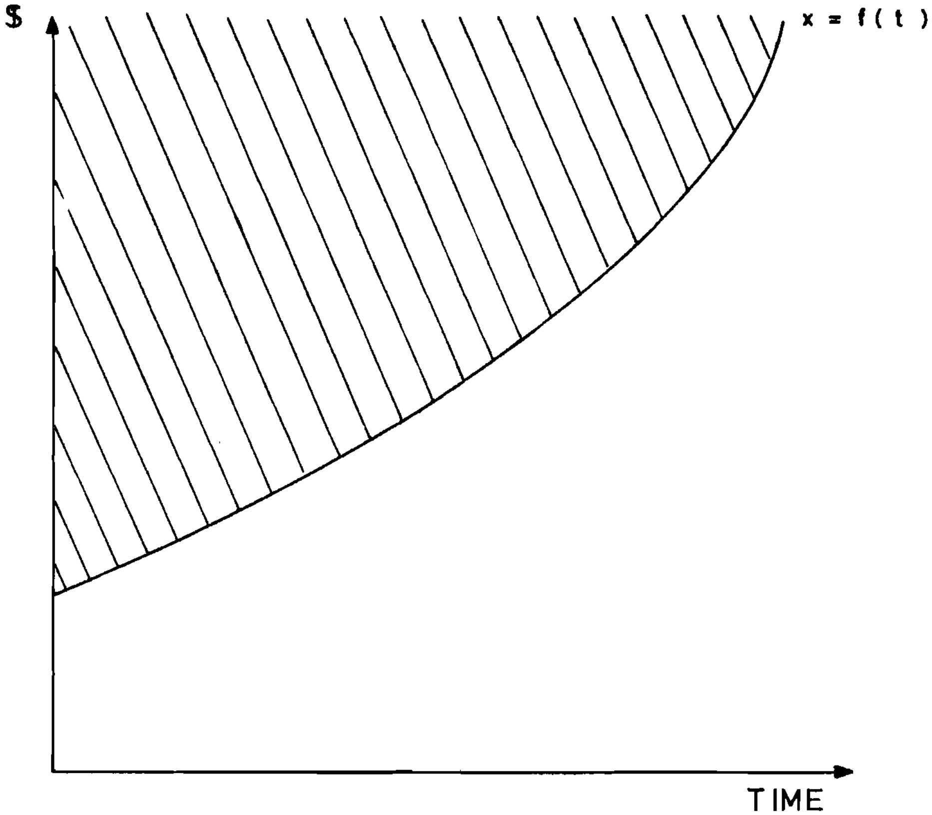

FIGURE 3. REGION IN WHICH THE UTILITY FUNCTION IS FULLY DETERMINED.

Koopmans [3,4] and Meyer [5], in particular, have discussed assumptions to simplify the problem of time stream utility evaluation. The solution proposed here, however, is based upon achieving an approximate solution, rather than an exact one based on simplifying assumptions.

We will consider the discrete case first, then deduce the analogous result in the continuous case. Consider again the stream $(x_0, x_1, x_2, \ldots)$ and three special examples of this stream, namely $(2,0,0,\ldots)$, $(0,2,0,0,\ldots)$, and $(2,2,0,0,0,\ldots)$. Let their utilities be u_1, u_2, and u_3 respectively. We can assert the following relations between these quantities:

i) $u_1 > u_2$ because of assumed impatience; it is preferable to receive money sooner rather than later.

ii) $u_3 < u_1 + u_2$, because in the two-income stream, the value of the second income is offset by the first.

So how may we judge the value of u_3? Consider the corresponding situation for the utility function for money alone, u^*. The utility of an income of x followed by one of y is $u^*(x + y)$ which may be written as

$$u^*(x) + [u^*(x + y) - u^*(x)] \quad ,$$

that is, the utility of the first income together with the increase in utility due to the second income. We will adopt this strategy in the case of our time streams and write

$$\begin{aligned}\bar{u}(2,2,0,0,\ldots) &= \bar{u}(2,0,0,\ldots) \\ &\quad + \bar{u}(0,4,0,0,\ldots) - \bar{u}(0,2,0,0,\ldots)\end{aligned}$$

or

$$\bar{u}(2,2,0,0,\ldots) = u(2,0) + u(4,1) - u(2,1) \quad .$$

In general, letting

$$X_k = \sum_{i=0}^{k} x_i$$

we have

$$\bar{u}(x_0,x_1,x_2,\ldots) = u(x_0,0) + \sum_{j=1}^{\infty} [u(X_j,j) - u(X_{j-1},j)] \tag{4}$$

as a measure of the utility of the time stream.

Note that if we return to the special case of discounting and substitute $u(x,t) = \dfrac{x}{(1 + r)^t}$ into (4) we obtain the required expression

$$\bar{u}(x_0,x_1,\ldots) = \sum_{i=0}^{\infty} \frac{x_i}{(1 + r)^i} \quad .$$

Transferring to the continuous form we have for a function x,

$$\bar{u}(x) = \int_0^{\infty} u[x(t),t] - u[x(t),\, t + \delta t]$$

$$= - \int_0^{\infty} \frac{\partial u[x(t),t]}{\partial t}\, dt \quad .$$

Rewriting equation (4) as

$$\sum_{j=0}^{\infty} [u(X_j,j) - u(X_j,j + 1)] \quad ,$$

an equivalent continuous formula is

$$\bar{u}(x) = \int_0^\infty \frac{\partial u(x,t)}{\partial x} \cdot \frac{dx}{dt} \cdot dt \quad .$$

Summary

The proposed system of dealing with the problem of delayed incomes (or payments), or time streams of such income, is complicated compared to the simplicity of fixed rate discounting. The aim of this paper however has been to present an alternate method which takes more account of the time preferences of the decision maker and yet is not intractable, for it is of little use presenting a perfect model which cannot be implemented.

If any of the three assumptions mentioned are felt to be applicable, so much the better, but the assessment of a two-dimensional utility function, whilst difficult, is not insuperable.

The increased accuracies gained from this system are twofold. Apart from the fact that the decision maker's time preferences have been represented by a two-dimensional function instead of a single constant r, there is also the advantage inherent in the use of utility functions. That is, in circumstances of risk and uncertainty in the quantity and timing of the incomes, often the case in business ventures, the utility function takes account of the decision maker's attitude towards risk taking.

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